

Lecture 6 Summary

PHYS798S Spring 2016

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The Cooper pairing problem, concluded

Go back to calculate the expansion coefficients for the ansatz wavefunction:

$$g_{\vec{k}} = \frac{V \sum_{\vec{k}_F < \vec{k}' < \vec{k}_c} g_{\vec{k}}}{2\varepsilon_k - E}, \text{ now substituting the eigen-energy gives,}$$
$$g_{\vec{k}} = \frac{V \sum_{\vec{k}_F < \vec{k}' < \vec{k}_c} g_{\vec{k}}}{2\xi_k + 2\hbar\omega_c e^{-2/D(E_F)V}}, \text{ where we define } \xi_k \equiv \varepsilon_k - E_F, \text{ which is the energy difference from the Fermi surface.}$$

Treating the sum in the numerator as just a number, which can be subsumed into the normalization factor, we get for the full wavefunction,

$$\Psi(1, 2) \propto \sum_{k > k_F} \frac{\cos(\vec{k} \cdot \vec{r})}{2\xi_k + 2\hbar\omega_c e^{-2/D(E_F)V}} |00\rangle.$$

Note two things about this expression. First there is no preferred direction in k -space, that is the solution wavefunction is isotropic in \vec{k} . This isotropy of the wavefunction is noted as "s-wave pairing" in the literature. Other orbital symmetries are possible and observed, including "p-wave" and "d-wave" pairing states. Second, the expansion coefficients decrease quickly with increasing distance from the Fermi surface, falling off like $1/k^2$. This justifies to some extent the simplified treatment of the pairing potential $V_{\vec{k}, \vec{k}}$.

0.1 Binding Energy Systematics

The binding energy depends on the energy scale of the ion vibrations $\hbar\omega_c$ as well as the strength of the pairing interaction V and the density of states (DOS) at the Fermi energy $D(E_F)$. Let's examine these dependencies, assuming that the superconducting transition temperature scales with the binding energy of the Cooper pair, i.e. $k_B T_c \sim 2\hbar\omega_c e^{-2/D(E_F)V}$.

It is found in the transition metals that T_c scales very closely with the DOS at the Fermi energy, $D(E_F)$. The Supplementary Material on the class web site shows this dependence. The oscillatory $D(E_F)$ variation (and T_c variation) upon going across the 4d transition metal alloys arises from the Fermi energy moving through peaks and valleys of the DOS in the fairly rigid band structure. We will study the dependence of T_c on $\hbar\omega_c$ after the discussion of the pairing interaction in the next lecture.

0.2 Estimating the Coherence Length

If we imagine that the Cooper pair is a minimum uncertainty wave packet, we can make an order of magnitude estimate of its spatial extent. Using the position momentum uncertainty relation $\Delta x \Delta p \sim \hbar$, and estimating $\Delta p \sim \frac{\hbar \omega_c e^{-2/D(E_F)V}}{v_F} \sim \frac{k_B T_c}{v_F}$, we find the typical size of a Cooper pair is,

$\Delta x \sim \frac{\hbar v_F}{k_B T_c}$. More correctly, BCS theory predicts this length scale is $\xi_0 = a \frac{\hbar v_F}{k_B T_c}$, where $a = 0.18$. Plugging in numbers for Al we find $\xi_0 = 0.18 \times 13.1 \mu m = 2.4 \mu m$. The accepted experimental value is about $1.6 \mu m$. Al has a relatively high Fermi velocity and low T_c , so its coherence length is rather large, putting it at the extreme end in fact. Most materials have coherence lengths ranging from about 1 to 100 nm. In Ginzburg-Landau theory we will interpret the coherence length as the characteristic length scale on which the superconducting order parameter can vary.

Going back to the Cooper pair size discussion, we see now that in a typical metal the pairs are highly overlapping. A Cooper pair of size 10^2 to 10^3 nm, with atoms of size 10^{-1} nm, will cover $(10^3)^3$ to $(10^4)^3$ or 10^9 to 10^{12} electrons! Hence each Cooper pair overlaps with "billions and billions" of others, making this very different from a Bosonic gas, such as 4He . The strongly overlapping nature of the pairs means that superconductivity is not simply a Bose-Einstein condensation of weakly interacting (spin-singlet) Bosons, but a phase transition of very different character.

Thinking ahead to the full BCS theory, we need to write down an ansatz wavefunction for ALL of the electrons in the metal. At this point it is clear that we are NOT able to write the full N -particle superconducting state wavefunction as $\Psi(1, 2, \dots, N) = \psi(1, 2)\psi(3, 4) \dots \psi(\frac{N}{2} - 1, \frac{N}{2})$. Something much more general that respects the Fermionic nature of the electrons is required.